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On a new integral formula for an invariant of 3-component oriented links ☆

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Abstract

Using physical considerations we constructed a new invariant of isotropy classes of an arbitrary configuration of three magnetic tubes in the space. The integral expression of this invariant is similar to the Massey product integrals of Milnor invariants of links. We prove that the constructed invariant cannot be expressed from the linking numbers of the configuration of magnetic tubes.

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We consider some new results towards the solution of the Problem by Arnol'd [1] [1990–16]. "What invariants of knots can be extended to invariants of divergence-free vector fields?" Note that this problem can be considered also for the case of links (multi-component knots) in $\mathbb{R}^3$. A similar problem [1984–12] is the following: "To transform the asymptotic ergodic definition of the Hopf invariant of a divergence-free vector field to the Novikov theory of generalized Whitehead products in homotopy groups." The most important case for applications is the three-dimensional case where the divergence-free vector field can represent

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a magnetic field or the vorticity field in fluid dynamics. In this case the generalized Whitehead product is called Massey product and these products express Milnor's invariants of multi-component links. A generalization to higher dimensions is considered by Khesin [2].

We will formulate a new problem in view of the solution of the Arnol'd–Novikov problem. Let us assume that a divergence-free vector field **B** is modelled by a link $L \subset \mathbb{R}^3$. This means that the support of the field coincides with a finite number of solid torus called magnetic tubes. Each tube U_i is equipped with the flux of the vector field **B** over a transversal cross-section of the tube. Inside each tube the field could have a very complex configuration, in particular, integral lines of the field could be non-compact. For a definition of a magnetic tube, see [3]. The decomposition of a magnetic field into tubes is not canonical. For example, two parallel tubes in space can be joined into one ambient tube and vice versa. In the considered tube U_i one could fix the central line.

Problem (A higher-order analog of the helicity integral). Let **B** be a divergence-free vector field decomposed into a finite number of magnetic tubes. The task is to find an integral expression of a higher invariant of **B** with respect to a volume-preserving diffeomorphism of the space with compact support that cannot be expressed from the linking numbers of pairs of the magnetic tubes.

This was an open problem before, because all known higher invariants for fields decomposed into tubes were not totally defined, but only partially defined. This means that an invariant is defined under the additional assumption that some of the more simple invariants of the field (e.g., linking numbers of pairs of tubes) are trivial. Therefore our result is of interest in topology. Using physical considerations we will construct a new invariant of isotopy classes of a three-component link with an integral expression similar to the Massey product integrals of Milnor invariants of links.

We consider briefly the contents of the paper. In Section 1 we recall the topological aspect of the problem and we recall some number of results toward the solution of the problem. We also formulate the main result. In Section 2 we consider the required preparation concerning the gauge of the potentials of the field decomposed into three ordered tubes. This consideration is based on the Milnor invariant of length 2 in the form presented in [29]. In Section 3 we present the integral invariant denoted by M. In Section 4 we prove the invariance of the integral formula with respect to gauge transformation of the potentials. This proves that M is an invariant with respect to volume-preserving diffeomorphisms of the space. In Section 5 we prove that M is non-degenerated and cannot be expressed from the linking coefficients. We also formulate an open problem.

1. Milnor invariants of multi-component oriented links and their integral expressions for magnetic fields

A multi-component oriented link in $\mathbb{R}^3$ is defined as a one-dimensional oriented smooth submanifold in $\mathbb{R}^3$ with ordered connected components. One can also determine such a link by means of a parameterization $f : S \subset \mathbb{R}^3$, where the parameter space consists of a collection $L = L_1, \dots, L_s$ of s standard circles. Milnor determined (see [4]) algebraic invariants of multi-component links called μ-invariants.

The simplest μ-invariant is the linking number $\mu_{i,j}$ of components L_i and L_j of a link L. The next invariant $\mu_{i,j,k}$ of length 2 determines a measure of the complexity of a link. This invariant is an integer under the assumption $\mu_{i,j} = 0$, $\mu_{j,k} = 0$, $\mu_{k,i} = 0$. In the general case $\mu_{i,j,k}$ is well defined (modulo($d(i, j, k)$), where $d(i, j, k)$ is a greatest common devisor of the integers $\mu_{i,j}$, $\mu_{j,k}$, $\mu_{k,i}$).

We recall that $\mu_{i,j,k}$ is defined as the coefficient in the decomposition of the element determined by the loop L_k in the fundamental group $\pi_1(\mathbb{R}^3 \setminus (L_i \cup L_j))$ over the basic commutators of length 2, see [4, p. 189; 5]. The three invariants $\mu_{i,j,k}$, $\mu_{j,k,i}$, $\mu_{k,i,j}$ are equal, nevertheless the definitions are different. The collection of the linking coefficients $\mu_{1,2}$, $\mu_{2,3}$, $\mu_{3,1}$ and the invariant $\mu_{1,2,3}$ determines a three-component link up to homotopy.

In the case $s = 4$ under the additional assumption $\mu_{i,j} = 0$, $\mu_{i,j,k} = 0$ (the link is called semi-boundary link if these conditions are satisfied) for an arbitrary order of the indices i, j, k, l the integer valued invariant $\mu_{i,j,k,l}$ is well defined. This invariant is defined by the coefficient in the decomposition of the element in the fundamental group $\pi_1(\mathbb{R}^3 \setminus (L_i \cup L_j \cup L_k))$ determined by the loop L_l over the basic commutators of length 3. This invariant depends on the order of the indices and we have 24 invariants, but only two of them are independent. These two invariants are called basic invariants. If we change the order of the indices, the new invariant can be expressed from the two basic invariants.

Two four-component semi-boundary links L, L' are homotopic if and only if the corresponding basic μ-invariants of length 3 are equal. Without the assumption that the link is semi-boundary the classification problem is very complicated (see [6]). The difficulty arises because the integer μ-invariants of length 3 are not well defined for an arbitrary four-component link.

Using the construction of the invariant $\mu_{i,j,k,l}$, the following invariants $\mu_{i,i,k,l}$, $\mu_{i,i,k,k}$ for three- and two-component links are defined correspondingly. To determine $\mu_{i,i,k,l}$, one uses a copy L_i' of the component L_i which is shifted in such a way that $lk(L_i; L_i') = 0$ and the invariants are defined by the formula

$$\mu_{i,i,k,l}(L_i, L_k, L_l) = \mu_{i,j,k,l}(L_i, L_j = L_i', L_k, L_l),$$
$$\mu_{i,i,k,k} = \mu_{i,j,k,l}(L_i, L_j = L_i', L_k, L_l = L_k').$$

The invariant $\mu_{i,i,k,k}$ is denoted by $\beta(i, k)$. This invariant, called the Sato–Levine invariant, was defined by a straightforward elementary construction in [7]. The invariant $\beta(i, k)$ of isotopy classes of links, generally speaking, is not well defined under a homotopy of links. One can show that the Sato–Levine invariant is preserved up to one quasi-isotopy of links, see [8].

An arbitrary invariant $\mu_{i_1,\ldots,i_s}$ admits an alternative description in terms of Massey products, see the papers by Turaev [9] and Porter [10]. The integral expressions for Massey invariants can be considered in the framework of magnetohydrodynamics (MHD). This was firstly shown in the paper [11] by Monastyrsky and Retakh, and also in [12–17]. Furthermore, all the invariants were investigated in the Ph.D. thesis by Mayer [18].

The integral expressions for Milnor invariants generalize the Gauss formula for the linking number of two closed oriented curves in $\mathbb{R}^3$, see [19]. The Gauss formula can be generalized to divergence-free vector fields; this generalization is called the helicity integral, see [20,3,18,21]. The helicity invariant was interpreted from the topological point of view in the paper by Arnol'd [22], see also [23]. In fluid dynamics the helicity integral is used as invariant of the vorticity field frozen into an ideal incompressible fluid and as an invariant of a frozen-in magnetic field, see [20,23]. A modification of the helicity invariant, the so-called cross-helicity of a vorticity field or a magnetic field was investigated in [24,25]. In topology the helicity integral is called the Hopf invariant or the Whitehead integral.

A topological invariant of a triple of divergence-free vector fields naturally appears when considering a Yang–Mills SU(2)-field. In the paper [17] the connection of the Chern–Simons three-form with the Milnor invariants of length 3 was found.

An important step towards the solution of the Arnol'd–Novikov problem is the following.

Problem. One has to describe various kinds of invariants of an oriented link that are defined without the additional assumption that the more simple invariants of its proper sublinks vanish.

We are looking for integral expressions for such invariants in the framework of MHD theory. From this point of view a decomposition of the magnetic field into tubes is feasible. The Sato–Levine invariant is an example of a Milnor invariant that can be naturally extended from two-component semi-boundary links (i.e. with vanishing lower linking coefficient) to an invariant for arbitrary two-component links. This invariant was discovered in the paper by Polyak and Viro [26] as a Vassiliev invariant of order 3. In the joint paper by the author with Malesic and Repovs it was proved that for an arbitrary two-component oriented semi-boundary link the Polyak–Viro invariant coincides with the Sato–Levine invariant, see [27,28]. Therefore this invariant is called a generalized Sato–Levine invariant. The generalized Sato–Levine invariant was discovered independently by Kirk and Livingston [30] and Repovs and the author in [29] by an elementary construction.

In the presentation by Malesic and Repovs at the conference "Knots in Poland", Warsaw (2003) it was shown that the properties of the generalized Sato–Levine invariant are analogous to the properties of the linking coefficient of the tubes, see also [31]. In particular, this gives a higher (non-linear) analog of the self-linking number of a closed tube that can be decomposed into the product of the Vassiliev invariant of order 2 and the self-linking number of the tube.

Unfortunately, the integral formula for the generalized Sato–Levine invariant does not appear naturally in MHD. Podkoritov showed to the author that the integral formula for the Sato–Levine invariant, discovered in [15,16], cannot be directly applied to a pair of magnetic fields.

1.1. The main result

For an arbitrary divergence-free vector field **B**, decomposed into three disjoint tubes, we construct an integral expression for an invariant M (see formula (20)). The expression is similar to the expression of Massey integrals for Milnor's invariants and we say that the invariant M is of Milnor type. The invariant M is of order 12. This means that the value M scales with λ^{12} under a change of the field **B** into λ**B**. We conjecture that the combinatorial formula for the invariant is the following:

$$M(1,2,3) = \mathbf{fl}_1^2\mathbf{fl}_2^2(1,3)^2(2,3)^2\beta_{1,2} + \mathbf{fl}_2^2\mathbf{fl}_3^2(1,2)^2(1,3)^2\beta_{2,3} + \mathbf{fl}_3^2\mathbf{fl}_1^2(1,2)^2(1,3)^2\beta_{3,1} + \mathbf{fl}_1^2\mathbf{fl}_2^2\mathbf{fl}_3^2(1,2)(2,3)(3,1)\gamma, \quad (1)$$

where the linking coefficient (i, j) is defined as $(i, j) = \mathbf{fl}_i\mathbf{fl}_j\mathbf{lk}_{i,j}$, where $\mathbf{fl}_i$, $\mathbf{fl}_j$ are the fluxes of the vector field $\mathbf{B}$ in the tubes U_i, U_j, respectively, $\mathbf{lk}_{i,j}$ is the linking number of the central lines of the tubes under consideration, $\beta(i, j)$ are the generalized Sato–Levine invariants of the two-component link presented by the central lines of the considered tubes, and γ is an invariant of three-component oriented links, the combinatorial expression of which is unknown.

We note that the generalized Sato–Levine invariant is not known in MHD. The order of invariants M, β, γ, in Vassiliev theory and in MHD is discussed in Remark 3.2.

From formula (20), obviously, one can deduce that the invariant M that was constructed for an ordered link (i.e. components of the link have to be equipped with integers 1–3) does not depend on the order of the components. This means that M is a well-defined invariant for fields decomposed into three tubes. In case of an arbitrary number of tubes (greater than 3), one can consider all possible triples of tubes and determines $M(\mathbf{B})$ as the sum of the values for the triples. Hence the construction of M provides a solution to the higher-order analog of the helicity integral problem formulated above.

2. An admissible gauge of potentials of the field decomposed into three disjoint tubes

Let the field $\mathbf{B}$ be decomposed into three tubes U_1, U_2, U_3 with central lines L_1, L_2, L_3, $L = L_1 \cup L_2 \cup L_3$. Let $\mathbf{fl}_1, \mathbf{fl}_2, \mathbf{fl}_3$ be the fluxes of $\mathbf{B}$ in the corresponding tubes, and let $\mathbf{lk}_{1,2}$, $\mathbf{lk}_{2,3}$, $\mathbf{lk}_{3,1}$ be the integer linking numbers of the corresponding central lines.

Consider a single tube U from the set of tubes U_1, U_2, U_3. We recall that a multivalued function $\xi : U \to \mathbb{R}$ is a function on the cyclic universal covering $\tilde{\xi} : \tilde{U} \to \mathbb{R}$ that satisfies the equation $\tilde{\xi} = \tilde{\xi} \simeq T + C$, where $T : \tilde{U} \to \tilde{U}$ is the shift of the cyclic covering with respect to the generator, the constant C in this formula is called the period of the multivalued function. This constant is determined by the equation

$$\oint_L \operatorname{grad} \xi \, \mathrm{d}s = C. \tag{2}$$

In particular, ξ is a function if and only if $C = 0$.

Let $L' \subset U$ be a central line of the tube U. We denote by $\tilde{x}_i$, $i \in \mathbb{Z}$ the set of inverse images of a point $x \in L'$ in the cyclic covering $\tilde{L}'$ over L'. The following sequence of values $\tilde{\xi}(\tilde{x}_i) = \tilde{y}_i$ is defined, and we have $\tilde{y}_{i+1} - \tilde{y}_i = C$.

We denote by $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ potentials for the fields $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$. We assume that the potentials $\mathcal{A}_i$ tend to 0 as $|x|^{-2}$ for $x \to \infty$. This condition ensures convergence of all the integrals under investigation. Let us consider the restriction of the potential $\mathcal{A}_i$ to the tube U_j, $i \neq j$. For an arbitrary pair $\{i, j\}$, $i \neq j$, $i = 1, 2, 3$, $j = 1, 2, 3$, we consider a multivalued function $\varphi_{i,j} : U_j \to \mathbb{R}$ subject to the equation

$$\operatorname{grad} \varphi_{i,j} = \mathcal{A}_i|_{U_j}. \tag{3}$$

We shall call the function $\varphi_{i,j}$ a branch of the potential $\mathcal{A}_i$ into the tube U_j. Such a branch $\varphi_{i,j}$ is defined by Eq. (3) up to a constant. The period of the branch $\varphi_{i,j}$ is equal to $\mathbf{lk}_{i,j}\mathbf{fl}_i$.

We describe the auxiliary integral expression α, that depends on potentials $\mathcal{A}_i$ and on branches $\varphi_{i,j}$ of the potentials. Let us consider a collection of embedded disks $\Gamma_i \subset U_i$, with the boundaries $\partial\Gamma_i$ embedded into the boundaries ∂U_i of the corresponded tubes. The homology class $[\Gamma_i, \partial\Gamma_i] \in H_2(U_i, \partial U_i; \mathbb{Z})$ is Poincare dual to the generator of the cohomology group $H^1(U_i; \mathbb{Z})$. The orientation of the disk Γ_i is determined such that the normal vector n at a point on this disk satisfies $\int(\mathbf{B}\cdot n)\,\mathrm{d}\Gamma_i > 0$. We shall call the disk Γ_i a cross-section disk in the tube U_i.

The given cross-section disks allow us to determine the following integral expressions:

$$I_1 = \int(\mathbf{B}_1, [\mathcal{A}_2\varphi_{3,1} - \mathcal{A}_3\varphi_{2,1}])\,\mathrm{d}U_1, \tag{4}$$

$$J_1 = \int(\mathbf{B}_1, n)[\varphi_{2,1}\mathbf{lk}_{3,1}\mathbf{fl}_3 - \varphi_{3,1}\mathbf{lk}_{2,1}\mathbf{fl}_2]\,\mathrm{d}\Gamma_1, \tag{5}$$

$$I_2 = \int(\mathbf{B}_2, [\mathcal{A}_3\varphi_{1,2} - \mathcal{A}_1\varphi_{3,2}])\,\mathrm{d}U_2, \tag{6}$$

$$J_2 = \int(\mathbf{B}_2, n)[\varphi_{3,2}\mathbf{lk}_{1,2}\mathbf{fl}_1 - \varphi_{1,2}\mathbf{lk}_{3,2}\mathbf{fl}_3]\,\mathrm{d}\Gamma_2, \tag{7}$$

$$I_3 = \int(\mathbf{B}_3, [\mathcal{A}_1\varphi_{2,3} - \mathcal{A}_2\varphi_{1,3}])\,\mathrm{d}U_3, \tag{8}$$

$$J_3 = \int(\mathbf{B}_3, n)[\varphi_{1,3}\mathbf{lk}_{2,3}\mathbf{fl}_2 - \varphi_{2,3}\mathbf{lk}_{3,1}\mathbf{fl}_1]\,\mathrm{d}\Gamma_3. \tag{9}$$

Let us describe the integral I_1 more precisely. Cut the tube U_1 along the surface Γ_1. The domain $U_1 \setminus \Gamma_1$ is homeomorphic to the standard ball such that two copies of the disk Γ_1 are embedded into the boundary $\partial(U_1 \setminus \Gamma_1)$ of this ball. We will denote these two disks by $\Gamma_{1,+}, \Gamma_{1,-}$. Let us consider the orientation of the disks. The disks are equipped with $\{-, +\}$ such that the positive normal vector over the disk $\Gamma_{1,-}$ points inside of the ball $U_1 \setminus \Gamma_1$, and the same normal vector over $\Gamma_{1,+}$ points outside of the ball. Let us consider branches $\varphi_{2,1}, \varphi_{3,1}$. We fix the set of the branches in the domain $U_1 \setminus \Gamma_1$ and, in particular, on the surface $\Gamma_{1,-} \subset \partial(U_1 \setminus \Gamma_1)$. We will denote the considered branches over $\Gamma_{1,-}$ by $\varphi_{\Gamma;2,1}$, $\varphi_{\Gamma;3,1}$ correspondingly.

The integral J_1 also depends on the choice of the disk Γ_1 and on a choice of the branches $\varphi_{2,1}, \varphi_{3,1}$ over this disk. This integral is determined as a surface integral of the product of the vector $\mathbf{B}$ with a function. In the integral expression we take the branches attached to the surface $\Gamma_{1,-}$. For the tubes U_2, U_3 the integrals (4)–(9) are given by an analogous expression.

Let us assume that the branches $\varphi_{2,1}, \varphi_{3,1}$ of the potentials $\mathcal{A}_2, \mathcal{A}_3$ over Γ_1 are fixed. Let us consider another cross-section disk Γ'_1 of U_1 and the restrictions $\varphi_{\Gamma';2,1}$, $\varphi_{\Gamma';3,1}$ of the branches $\varphi_{2,1}, \varphi_{3,1}$ correspondingly. This means that there exists an isotopy in the space of cross-section disks from the disk Γ to the disk Γ' that induces a transformation of the function $\varphi_{\Gamma;2,1}$ of the branch $\varphi_{2,1}$ over Γ to the function $\varphi_{\Gamma';2,1}$ of the same branch over Γ'. Simultaneously, the same isotopy induces the transformation from the function $\varphi_{\Gamma;3,1}$ of the branch $\varphi_{3,1}$ over Γ to the function $\varphi_{\Gamma';3,1}$ of the branch $\varphi_{3,1}$ over Γ'.

The difference $I_1 - J_1$ does not depend on the choice of Γ_1. This means that this difference is not changed if we replace arbitrary values of the pair of the branches over the disk Γ_1 to the corresponding pair of the values of the same branches over the disk Γ'_1. Let us formulate the following lemma.

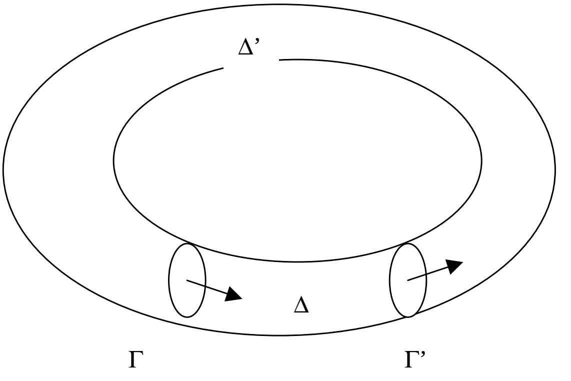

Fig. 1. Integral $I_1 - J_1$ is well defined.

Lemma 2.1. *The integral $I_1 - J_1$, defined by Eqs. (4) and (5) is not changed if we change the cross-section disk Γ_1 to a cross-section disk Γ'_1. The analogous rule holds for the integrals $I_2 - J_2$, $I_3 - J_3$, determined by formulas (6)–(9).*

Proof of Lemma 2.1. We prove the lemma for the integral $I_1 - J_1$. For briefness, we will drop the subscripts on the disks Γ_1 and Γ'_1. Let us assume that the disks Γ, Γ' do not intersect. We denote by $\Delta \subset U_1$ the domain bounded by the disks Γ, Γ' and by a part of the boundary ∂U_1 of the tube. We have the two possible parts of ∂U_1 and we take the part bounded by the disk $\Gamma_- \subset \partial\Delta$ with the interior normal vector and by the disk Γ'_+ with the exterior normal vector. Let us consider the expressions $I_1(\Gamma) - I_1(\Gamma')$, $J_1(\Gamma) - J_1(\Gamma')$. We will show that $I_1(\Gamma) - J_1(\Gamma) = I_1(\Gamma') - J_1(\Gamma')$ (Fig. 1).

Let us consider the simplest case. We assume that the values $\varphi_{\Gamma';i,1}$, $i = 2, 3$ over the surfaces Γ' are obtained by the extension over Δ of the branches $\varphi_{\Gamma;i,1}$ of the potential $\varphi_{i,1}$. In this case the difference $I_1(\Gamma') - I_1(\Gamma)$ is given by the formula

$$I_1(\Gamma') - I_1(\Gamma) = \int (\mathbf{B}_1, [\mathcal{A}_2(\varphi_{3,1} + C_3) - \mathcal{A}_3(\varphi_{2,1} + C_2)])\,\mathrm{d}\Delta - \int (\mathbf{B}_1, [\mathcal{A}_2\varphi_{3,1} - \mathcal{A}_3\varphi_{2,1}])\,\mathrm{d}\Delta,$$

where C_3 is the period of the branch $\varphi_{3,1}$, C_2 the period of the branch $\varphi_{2,1}$. To prove this, we observe that in the complimentary domain Δ', $\Delta' = U_1 \setminus \Delta$, the functions in the integrals I_1 and I'_1 are equal, but in the domain Δ the corresponding branches in the integral differ by the period. Because of the equation $C_2 = \mathbf{lk}_{1,2}\mathbf{fl}_2$, $C_3 = \mathbf{lk}_{3,1}\mathbf{fl}_3$, the expression reduces to

$$I_1(\Gamma') - I_1(\Gamma) = \int (\mathbf{B}_1, \mathcal{A}_2)\mathbf{lk}_{3,1}\mathbf{fl}_3 - (\mathbf{B}_1, \mathcal{A}_3)\mathbf{lk}_{2,1}\mathbf{fl}_2\,\mathrm{d}\Delta.$$

Note that $\mathrm{div}[\mathbf{B}_1\varphi_{2,1}\mathbf{lk}_{3,1}\mathbf{fl}_3 - \mathbf{B}_1\varphi_{3,1}\mathbf{lk}_{2,1}\mathbf{fl}_2] = (\mathbf{B}_1, \mathcal{A}_2)\mathbf{lk}_{3,1}\mathbf{fl}_3 - (\mathbf{B}_1\mathcal{A}_3)\mathbf{lk}_{2,1}\mathbf{fl}_2$.

By the Gauss–Ostrogradsky formula in the domain Δ we obtain

$$I_1(\Gamma') - I_1(\Gamma) = \int (\mathbf{B}_1, n)[\varphi_{2,1}\mathbf{lk}_{3,1}\mathbf{fl}_3 - \varphi_{3,1}\mathbf{lk}_{2,1}\mathbf{fl}_2]\,\mathrm{d}\Gamma' - \int (B_1, n)[\varphi_{2,1}\mathbf{lk}_{3,1}\mathbf{fl}_3 - \varphi_{3,1}\mathbf{lk}_{1,2}\mathbf{fl}_2]\,\mathrm{d}\Gamma.$$

Obviously, the integral $J_1(\Gamma') - J_1(\Gamma)$ is the same.

Let us consider a more general case. We assume that the cross-section disks Γ, Γ' are disjoint, but the corresponding branches are not extended over the domain Δ. The calculation of the previous case can be done directly on the universal covering $\tilde{\Gamma}$ and the domain $\tilde{\Delta} \subset \tilde{\Gamma}$ between the disks.

The general case $\Gamma \cap \Gamma' \neq \emptyset$ is reduced to the previous case, because an arbitrary isotopy can be decomposed into a sequence of isotopies that joins the disks $\Gamma_1 = \Gamma$ and $\Gamma' = \Gamma_k$ by a sequence of disjoint disks Γ_i, $i = 1, \ldots, k$, and $\Gamma_j \cap \Gamma_{j+1} = \emptyset$, $j = 1, \ldots, k$. Lemma 2.1 is proved. □

Definition 2.2. Let $\varphi_{i,j}$ be a branch of potential A_i in the tube U_j. We define the real number α by the following formula:

$$\alpha(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3; \{\varphi_{i,j}\}) = 2\int \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\rangle\,\mathrm{d}\mathbb{R}^3 + I_1 - J_1 + I_2 - J_2 + I_3 - J_3. \tag{10}$$

In the case $\mathbf{lk}_{1,2} = \mathbf{lk}_{2,3} = \mathbf{lk}_{3,1} = 0$, we have α is a well-defined invariant of the ordered triple tubes $\{U_1, U_2, U_3\}$ with the last term in the form described in [16] (see also [17]).

Let us assume up to the end of this section that the following equation holds:

$$\mathbf{lk}_{1,2}^2 + \mathbf{lk}_{2,3}^2 + \mathbf{lk}_{3,1}^2 \neq 0, \tag{11}$$

i.e. there exist a nontrivial linking number between tubes $\{U_1, U_2, U_3\}$. In this case α in Eq. (10) is not well defined and depends on a choice of the branches $\varphi_{i,j}$.

Definition 2.3. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be potentials of the fields in the corresponding tubes. The collection $\varphi_{i,j}$, $i \neq j$ of branches of the potentials $\mathcal{A}_i$ is called an admissible collection if the following condition holds:

$$\alpha(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3; \{\varphi_{i,j}\}) = 0. \tag{12}$$

Lemma 2.4. *Let a collection of branches $\varphi_{i,j}$ of potentials $\mathcal{A}_i$ be admissible. Let us consider the gauge transformation $\mathcal{A}'_i = \mathcal{A}_i + \operatorname{grad} f_i$, where f_i are arbitrary functions over $\mathbb{R}^3$. (We do not assume that $f(x) \to 0$, if $x \to \infty$. We only assume that* $\operatorname{grad} f_i \to 0$ *as* $\|x\|^{-2}$.) *Then the collection of branches of the potentials $\mathcal{A}'_i$, given by the formula*

$$\varphi'_{i,j} = \varphi_{i,j} + f_i|_{U_j}, \quad i \neq j, \tag{13}$$

is also admissible.

Lemma 2.5. *Let a collection of branches $\varphi_{i,j}$ of potentials $\mathcal{A}_i$ be admissible. Let us consider the gauge transformation for the branches given by the following formula*:

$$\varphi_{i,j} \rightsquigarrow \varphi_{i,j} + \mathbf{fl}_i \mathbf{lk}_{k,i} C, \qquad \varphi_{j,i} \to \varphi_{j,i} + \mathbf{fl}_j \mathbf{lk}_{k,j} C. \tag{14}$$

Then the collection of the branches $\varphi'_{i,j}$ is also admissible.

Remark. The transformation (15) remains the potentials $\mathcal{A}_i$ fixed and, generally speaking, cannot be deduced from the transformation (14).

Lemma 2.6. *Let two collections of branches $\varphi_{i,j}$, $\varphi'_{i,j}$ of potentials $\mathcal{A}_i$, $\mathcal{A}'_i$ correspondingly, be admissible. Then the two collections of the branches $\{\varphi_{i,j}\}$ and $\{\varphi'_{i,j}\}$ are joined by the sequence of the transformations* (13) *and* (14) *described in* Lemmas 2.4 *and* 2.5.

Proof of Lemma 2.4. Let us consider a gauge transformation

$$\mathcal{A}_1 \rightsquigarrow \mathcal{A}_1 + \operatorname{grad} f,$$

the case of transformations of the potentials $\mathcal{A}_2, \mathcal{A}_3$ is analogous. Let us denote the restrictions of the function f to the tubes U_2, U_3 by f_2 and by f_3, respectively. In this case we have $\varphi_{1,2} \rightsquigarrow \varphi_{1,2} + f_2$, $\varphi_{1,3} \rightsquigarrow \varphi_{1,3} + f_3$. The other branches of the potentials $\psi_{i,j}$, $i = 2, 3$, $i \neq j$ are not changed. Under this gauge transformation we have

$$\alpha \rightsquigarrow \alpha + 2\int (\operatorname{grad} f, \mathcal{A}_2 \times \mathcal{A}_3)\, \mathrm{d}\mathbb{R}^3 + \delta I_2 - \delta J_2 + \delta I_3 - \delta J_3, \tag{15}$$

where

$$\delta I_2 = \int [-(\mathbf{B}_2, \operatorname{grad} f_2)\varphi_{3,2} + (\mathbf{B}_2, \mathcal{A}_3) f_2]\, \mathrm{d}U_2,$$

$$\delta I_3 = \int [(\mathbf{B}_3, \operatorname{grad} f_3)\varphi_{2,3} - (\mathbf{B}_3, \mathcal{A}_2) f_3]\, \mathrm{d}U_3,$$

$$\delta J_2 = -\int (\mathbf{B}_2, n) f_2 \mathbf{lk}_{2,3} \mathbf{fl}_3\, \mathrm{d}\Gamma_2, \qquad \delta J_3 = \int (\mathbf{B}_3, n) f_3 \mathbf{lk}_{2,3} \mathbf{fl}_2\, \mathrm{d}\Gamma_3.$$

Because $\int (\mathbf{B}_2, \mathcal{A}_3) f_2\, \mathrm{d}U_2 - \int (\mathbf{B}_2, \operatorname{grad} f_2)\varphi_{3,2}\, \mathrm{d}U_2 - \int (\mathbf{B}_2, n) f_2 \mathbf{lk}_{2,3} \mathbf{fl}_3\, \mathrm{d}\Gamma_2 = 0$, the term $\delta I_2 - \delta J_2$ in the gauge transformation of the integral is given by $2\int (\mathbf{B}_2, \mathcal{A}_3) f_2\, \mathrm{d}U_2$. The gauge transformation of the integral $I_3 - J_3$ is given by $-2\int (\mathbf{B}_3, \mathcal{A}_2) f_3\, \mathrm{d}U_3$. Note that the main term in the expression (15) is simplified by means of the Gauss–Ostrogradsky formula as follows: $2\int (\operatorname{grad} f, \mathcal{A}_2 \times \mathcal{A}_3)\, \mathrm{d}\mathbb{R}^3 = -2\int (\mathbf{B}_2, \mathcal{A}_3) f_2\, \mathrm{d}U_2 + 2\int (\mathbf{B}_3, \mathcal{A}_2) f_3\, \mathrm{d}U_3$. Therefore the considered gauge transformation leaves α unchanged. Lemma 2.4 is proved. □

Proof of Lemma 2.5. Let us prove that the transformation (13) does not change the sum of the last terms. Let us consider the case of a transformation $\varphi_{1,2} \rightsquigarrow \varphi_{1,2} + \mathbf{lk}_{3,1} \mathbf{fl}_1 C$, $\varphi_{2,1} \rightsquigarrow \varphi_{2,1} + \mathbf{lk}_{2,3} \mathbf{fl}_2 C$. Obviously, the integrals I_3, J_3 are not changed.

By formulae (4) and (5) we obtain

$$I_1 \rightsquigarrow I_1 + \mathbf{lk}_{2,3}\mathbf{fl}_2 C \int (\mathbf{B}_1, \mathcal{A}_3)\,\mathrm{d}U_1 = I_1 - \mathbf{fl}_1\mathbf{fl}_2\mathbf{fl}_3\mathbf{lk}_{2,3}\mathbf{lk}_{3,1}C,$$

$$-J_1 \to -J_1 - \mathbf{lk}_{2,3}\mathbf{lk}_{3,1}\mathbf{fl}_2\mathbf{fl}_3 C \int (B_1, n)\,\mathrm{d}\Gamma_1 = -J_1 - \mathbf{lk}_{2,3}\mathbf{lk}_{3,1}\mathbf{fl}_1\mathbf{fl}_2\mathbf{fl}_3 C,$$

$$I_2 \rightsquigarrow I_2 + \mathbf{lk}_{3,1}\mathbf{fl}_1 C \int (\mathbf{B}_2, \mathcal{A}_3)\,\mathrm{d}U_2 = I_2 + \mathbf{fl}_1\mathbf{fl}_2\mathbf{fl}_3\mathbf{lk}_{2,3}\mathbf{lk}_{3,1}C,$$

$$-J_2 \to -J_2 + \mathbf{lk}_{2,3}\mathbf{lk}_{3,1}\mathbf{fl}_1\mathbf{fl}_3 C \int (B_2, n)\,\mathrm{d}\Gamma_2 = -J_2 + \mathbf{lk}_{2,3}\mathbf{lk}_{3,1}\mathbf{fl}_1\mathbf{fl}_2\mathbf{fl}_3 C.$$

This proves that the gauge transformation preserves the integral $I_1 - J_1 + I_2 - J_2$. Lemma 2.4 is proved. □

Proof of Lemma 2.6. Two potentials $\mathcal{A}_i, \mathcal{A}'_i$ are related by the equation $\mathcal{A}_i - \mathcal{A}'_i = \operatorname{grad} f$, where f is a function over $\mathbb{R}^3$, $f(x) \to 0, x \to \infty$. We can apply a gauge transformation (13) from Lemma 2.4, therefore without loss of generality we may assume that the corresponding potentials $\mathcal{A}_i, \mathcal{A}'_i$ are equal. It is sufficient to prove that an arbitrary gauge transformation $\varphi_{i,j} \rightsquigarrow \varphi_{i,j} + C_{i,j}$ (where $C_{i,j}$ is a collection of constants) that transforms an admissible collection of branches to another admissible collection (i.e. keeps the value α) is decomposed into a finite sequence of transformations (13) for the special case $f_i =$ const. and transformations (14).

We start with the simplest case and we assume that two linking numbers are trivial, namely $\mathbf{lk}_{2,3} = \mathbf{lk}_{3,1} = 0$, $\mathbf{lk}_{1,2} \neq 0$. Using a transformation (13) for the potential A_1 (A_2) with $f =$ const. and (14) for $i = 1$, $j = 3$ ($i = 2$, $j = 3$) we obtain the branches $\varphi_{1,2}, \varphi_{2,3}$ with the required conditions. Using a transformation (14) with $f =$ const. we transform the branch $\varphi_{3,1}$ to the required branch $\varphi'_{3,1}$. Because of the assumption of admissibility of the collection of branches we also have $\varphi_{3,2} = \varphi'_{3,2}$. This proves Lemma 2.6 in this simplest case.

We consider the case $\mathbf{lk}_{1,2} \neq 0$, $\mathbf{lk}_{3,1} \neq 0$, $\mathbf{lk}_{2,3} = 0$.

Let us consider the tube U_1 and the branches $\varphi_{1,2}, \varphi_{1,3}$. Put $\varphi_{i,j} + C_{i,j} = \varphi'_{i,j}$. Using a transformation (14) we transform both the branches $\varphi_{1,2}$ and $\varphi_{1,3}$ to the required branches $\varphi'_{1,2}, \varphi'_{1,3}$. Then, using a transformation (13), we transform the branches $\varphi_{2,1}$ and $\varphi_{3,1}$ to the required branches $\varphi'_{2,1}$ and $\varphi'_{3,1}$. Now, using a transformation (14) we transform the pair of branches $\varphi_{2,3}$ and $\varphi_{3,2}$ such that $\varphi_{2,3} = \varphi'_{2,3}$. But in this case we also have $\varphi_{3,2} = \varphi'_{3,2}$, otherwise this contradicts with the assumption of admissibility of the collection of the branches $\varphi_{i,j}, \varphi'_{i,j}$. In this particular case Lemma 2.6 is proved.

Let us consider the general case $\mathbf{lk}_{1,2}\mathbf{lk}_{2,3}\mathbf{lk}_{3,1} \neq 0$. Started with a transformation (13), we transform the branches $\varphi_{1,2}, \varphi_{2,3}, \varphi_{3,1}$ to the required branches $\varphi'_{1,2}, \varphi'_{2,3}, \varphi'_{3,1}$ correspondingly. We obtain the condition $\varphi_{2,1} = \varphi'_{2,1}$. Using a transformation (14) for the pair of the branches $\varphi_{2,3}, \varphi_{3,2}$, we obtain the condition $\varphi_{2,3} = \varphi'_{2,3}$. Analogously, we obtain the condition $\varphi_{3,2} = \varphi'_{3,2}$. For the last two branches we also have $\varphi_{3,2} = \varphi'_{3,2}$, otherwise this contradicts with the assumption of admissibility of the two collection of the branches. Thus Lemma 2.6 is proved. □

3. The integral expression for the M invariant

Let us consider three tubes $U_1, U_2, U_3 \subset \mathbb{R}^3$ with central lines L_1, L_2, L_3. Let $\mathbf{B}$ be a divergence-free vector field decomposed into three fields $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ the supports of which coincide with these tubes. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be the corresponding potentials, i.e. $\operatorname{rot}\mathcal{A}_i = \mathbf{B}_i$, $i = 1, 2, 3$, $\mathcal{A}_i \to 0$, $x \to \infty$.

We recall that the field $\mathcal{A}_i$ outside the tube U_i is given by a gradient of a multivalued function denoted by φ_i. We have $\oint \mathcal{A}_i \,\mathrm{d}C_i = \mathbf{fl}_i$, where C_i is the boundary of a cross-section disk for the tube U_i. We denote by (i, j) the linking coefficient of the tubes U_i, U_j given by $\int A_i B_j \,\mathrm{d}U_j = \int A_j B_i \,\mathrm{d}U_i$. This coefficient is expressed from the linking number $\mathbf{lk}_{i,j}$ of the central lines of the tubes by the formula $(i, j) = \mathbf{fl}_i \mathbf{fl}_j \mathbf{lk}_{i,j}$.

Let $\varphi_{i,j}$ be a branch of the potential $\mathcal{A}_i$ in the tube U_j. Let us consider the following linear combinations of the multivalued functions:

$$\Phi_1 = (3, 1)\varphi_{2,1} - (1, 2)\varphi_{3,1}, \qquad \Phi_1 : U_1 \to \mathbb{R}^1, \tag{16}$$

$$\Phi_2 = (1, 2)\varphi_{3,2} - (2, 3)\varphi_{1,2}, \qquad \Phi_2 : U_2 \to \mathbb{R}^1, \tag{17}$$

$$\Phi_3 = (2, 3)\varphi_{1,3} - (3, 1)\varphi_{2,3}, \qquad \Phi_3 : U_3 \to \mathbb{R}^1. \tag{18}$$

The multivalued functions Φ_i have trivial period and therefore are single-valued functions. We define the vector field $\mathbf{F}$ by the formula

$$\begin{aligned}\mathbf{F} = {} & (2, 3)(3, 1)\mathcal{A}_1 \times \mathcal{A}_2 + (3, 1)(1, 2)\mathcal{A}_2 \times \mathcal{A}_3 + (1, 2)(2, 3)\mathcal{A}_3 \times \mathcal{A}_1 \\ & - (2, 3)\Phi_1\mathbf{B}_1 - (3, 1)\Phi_2\mathbf{B}_2 - (1, 2)\Phi_3\mathbf{B}_3.\end{aligned} \tag{19}$$

The following calculation shows that $\mathbf{F}$ is divergence-free:

$$\begin{aligned}& \operatorname{div}[(2, 3)(3, 1)\mathcal{A}_1 \times \mathcal{A}_2 + (3, 1)(1, 2)\mathcal{A}_2 \times \mathcal{A}_3 + (1, 2)(2, 3)\mathcal{A}_3 \times \mathcal{A}_1] \\ & \quad = (2, 3)(3, 1)[(\mathbf{B}_1, \mathcal{A}_2) - (\mathcal{A}_1, \mathbf{B}_2)] + (3, 1)(1, 2)[(\mathbf{B}_2, \mathcal{A}_3) - (\mathcal{A}_2, \mathbf{B}_3)] \\ & \qquad + (1, 2)(2, 3)[(\mathbf{B}_3, \mathcal{A}_1) - (\mathcal{A}_3, \mathbf{B}_1)] \\ & \quad = (2, 3)[(1, 3)(\mathbf{B}_1, \mathcal{A}_2) - (1, 2)(\mathbf{B}_1, \mathcal{A}_3)] + (3, 1)[(1, 2)(\mathbf{B}_2, \mathcal{A}_3) \\ & \qquad - (2, 3)(\mathbf{B}_2, \mathcal{A}_1)] + (1, 2)[(2, 3)(\mathbf{B}_3, \mathcal{A}_1) - (3, 1)(\mathbf{B}_3, \mathcal{A}_2)] \\ & \quad = (\mathbf{B}_1, (2, 3)\operatorname{grad}\Phi_1) + (\mathbf{B}_2, (1, 3)\operatorname{grad}\Phi_2) + (\mathbf{B}_3, (1, 2)\operatorname{grad}\Phi_3).\end{aligned}$$

Therefore there exist a vector potential $\mathbf{G}$, $\mathbf{G} \to 0$, $x \to \infty$, such that $\operatorname{rot}(\mathbf{G}) = \mathbf{F}$.

Let us assume that the collection of branches $\varphi_{i,j}$ in formulae (16)–(18) is admissible (see Definition 2.3 and Eq. (12)). The M invariant is defined by the following integral expression:

$$\begin{aligned}M(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) = {} & \int \mathbf{G}\mathbf{F}\,\mathrm{d}\mathbb{R}^3 - (2, 3)^2 \int \Phi_1^2(\mathcal{A}_1, \mathbf{B}_1)\,\mathrm{d}U_1 \\ & - (3, 1)^2 \int \Phi_2^2(\mathcal{A}_2, \mathbf{B}_2)\,\mathrm{d}U_2 - (1, 2)^2 \int \Phi_3^3(\mathcal{A}_3, \mathbf{B}_3)\,\mathrm{d}U_3\end{aligned}$$

$$+(2,3)(1,2)\int(\mathcal{A}_3,\mathbf{B}_1)\Phi_1^2\,\mathrm{d}U_1+(3,1)(2,3)\int(\mathcal{A}_1,\mathbf{B}_2)\Phi_2^2\,\mathrm{d}U_2$$
$$+(1,2)(3,1)\int(\mathcal{A}_2,B_3)\Phi_3^2\,\mathrm{d}U_3. \tag{20}$$

The symmetry of this expression can be shown by making use of the following lemma.

Lemma 3.1. *The following equations hold*:

$$(1,2)\int(\mathcal{A}_3,B_1)\Phi_1^2\,\mathrm{d}U_1=(3,1)\int(\mathcal{A}_2,\mathbf{B}_1)\Phi_1^2\,\mathrm{d}U_1, \tag{21}$$

$$(2,3)\int(\mathcal{A}_1,B_2)\Phi_2^2\,\mathrm{d}U_2=(1,2)\int(\mathcal{A}_3,\mathbf{B}_2)\Phi_2^2\,\mathrm{d}U_2, \tag{22}$$

$$(3,1)\int(\mathcal{A}_2,B_3)\Phi_3^2\,\mathrm{d}U_3=(2,3)\int(\mathcal{A}_1,\mathbf{B}_3)\Phi_3^2\,\mathrm{d}U_3. \tag{23}$$

Proof of Lemma 3.1. Let us prove Eq. (23), the other two equations are similar. We have $\int(((2,3)\mathcal{A}_1-(3,1)\mathcal{A}_2),\mathbf{B}_3)\Phi_3^2\,\mathrm{d}U_3=\int(\mathrm{grad}\,\Phi_3\Phi_3^2,\mathbf{B}_3)\,\mathrm{d}U_3=(1/3)\int\mathrm{div}(\Phi_3^3\mathbf{B}_3)\,\mathrm{d}U_3=0$. Lemma 3.1 is proved. □

Remark 3.2. The expression (20) is of order 12. This means that under the transformation $\mathbf{B}\rightsquigarrow\lambda\mathbf{B}$, $\lambda\in\mathbb{R}$, expression (20) transforms as $M\rightsquigarrow\lambda^{12}M$, because $(i,j)\rightsquigarrow\lambda^2(i,j)$, $\mathbf{F}\rightsquigarrow\lambda^6\mathbf{F}$, $\mathbf{G}\rightsquigarrow\lambda^6\mathbf{G}$.

We can also assume that the order of M in the meaning of Vassiliev is equal to 7, because the order of the Sato–Levine invariants $\beta_{i,j}$ is equal to 3. In this case the order of γ (see Eq. (1)) in the meaning of Vassiliev has to be 4.

4. Gauge transformations

To prove that M is a well-defined invariant with respect to volume-preserving diffeomorphisms with compact support, it is sufficient to prove that M is not changed under gauge transformations described in Lemmas 2.4 and 2.5. Without loss of generality one can assume that Eq. (11) holds, i.e. the case $\mathbf{lk}_{1,2}=\mathbf{lk}_{2,3}=\mathbf{lk}_{3,1}=0$ may be omitted. This allows as to assume that the branches of the potential are admissible (we may assume that at least two of the three linking numbers are nontrivial) since otherwise the invariant M is equal to 0.

Let us consider the following gauge transformation:

$$\mathcal{A}_1\rightsquigarrow\mathcal{A}_1+\mathrm{grad}\,f, \tag{24}$$

and the corresponding transformation (13) for the potentials. The induced transformation of the branches of the potentials was described in Lemma 2.4. Let us denote the restriction $f|_{U_i}$ by f_i. In particular, inside the tubes U_2, U_3, the gauge transformation of the branches of the potentials is given by

$$\varphi_{1,i}\rightsquigarrow\varphi_{1,i}+f_i.$$

This results in the following change of the vector **F**:

$$\mathbf{F} \rightsquigarrow \mathbf{F} + (2,3)(3,1)\,\mathrm{rot}(f\mathcal{A}_2) - (1,2)(2,3)\,\mathrm{rot}(f\mathcal{A}_3).$$

Without loss of generality we can assume that the additional term is small, and we calculate the first-order term in the gauge transformation of expression (20).

Let us briefly prove that the gauge transformation preserves M. The transformation of the main three terms of $\int(\delta G, F)\,\mathrm{d}\mathbb{R}^3$ is trivial by algebraic reason. The last term of the previous integral is cancelled with the extra summand in (20) by the argument from [16]. Let us present more detailed calculations:

$$M \rightsquigarrow M - 2\int (2,3)(3,1)(f\mathcal{A}_2, [(2,3)\Phi_1\mathbf{B}_1 + (3,1)\Phi_2\mathbf{B}_2 + (1,2)\Phi_3\mathbf{B}_3])\,\mathrm{d}\mathbb{R}^3$$
$$+ 2\int (1,2)(2,3)(f\mathcal{A}_3, [(2,3)\Phi_1\mathbf{B}_1 + (3,1)\Phi_2\mathbf{B}_2 + (1,2)\Phi_3\mathbf{B}_3])\,\mathrm{d}\mathbb{R}^3.$$

The additional gauge term in the expression can be presented by the following sum of the six terms, given by (25)–(27):

$$-2(2,3)(3,1)^2\int \Phi_2 f(\mathcal{A}_2, \mathbf{B}_2)\,\mathrm{d}U_2 + 2(1,2)^2(2,3)\int \Phi_3 f(\mathcal{A}_3, \mathbf{B}_3)\,\mathrm{d}U_3, \tag{25}$$

$$-2(3,1)(2,3)^2\int \Phi_1 f(\mathcal{A}_2, \mathbf{B}_1)\,\mathrm{d}U_1 + 2(1,2)(2,3)^2\int \Phi_1 f(\mathcal{A}_3, \mathbf{B}_1)\,\mathrm{d}U_1, \tag{26}$$

$$-\,2(1,2)(2,3)(3,1)\int \Phi_3 f(\mathcal{A}_2, \mathbf{B}_3)\,\mathrm{d}U_3 + 2(1,2)(2,3)(3,1)\int \Phi_2 f(\mathcal{A}_3, \mathbf{B}_2)\,\mathrm{d}U_2. \tag{27}$$

The last term in (20) is transformed by the following equations ((28)–(33)):

$$-(2,3)^2\int \Phi_1^2(\mathcal{A}_1, \mathbf{B}_1)\,\mathrm{d}U_1 \rightsquigarrow -(2,3)^2\int \Phi_1^2(\mathcal{A}_1, \mathbf{B}_1)\,\mathrm{d}U_1$$
$$-\,(2,3)^2\int \Phi_1^2(\mathrm{grad}\, f, \mathbf{B}_1)\,\mathrm{d}U_1, \tag{28}$$

$$-(3,1)^2\int \Phi_2^2(\mathcal{A}_2, \mathbf{B}_2)\,\mathrm{d}U_2 \rightsquigarrow -(3,1)^2\int \Phi_2^2(\mathcal{A}_2, \mathbf{B}_2)\,\mathrm{d}U_2$$
$$+\,2(3,1)^2\int \Phi_2(2,3) f(\mathcal{A}_2, \mathbf{B}_2)\,\mathrm{d}U_2, \tag{29}$$

$$-(1,2)^2\int \Phi_3^2(\mathcal{A}_3, \mathbf{B}_3)\,\mathrm{d}U_3 \rightsquigarrow -(1,2)^2\int_{U_3} \Phi_3^2(\mathcal{A}_3, \mathbf{B}_3)\,\mathrm{d}U_3$$
$$-\,2(1,2)^2\int \Phi_3 f(2,3)(\mathcal{A}_3, \mathbf{B}_3)\,\mathrm{d}U_3, \tag{30}$$

$$(1,2)(2,3)\int \Phi_1^2(\mathcal{A}_3,\mathbf{B}_1)\,\mathrm{d}U_1 \rightsquigarrow (1,2)(2,3)\int \Phi_1^2(\mathcal{A}_3,\mathbf{B}_1)\,\mathrm{d}U_1, \tag{31}$$

$$(1,2)(3,1)\int \Phi_3^2(\mathcal{A}_2,\mathbf{B}_3)\,\mathrm{d}U_3 \rightsquigarrow (1,2)(3,1)\int \Phi_3^2(\mathcal{A}_2,\mathbf{B}_3)\,\mathrm{d}U_3 + 2(1,2)(1,3)(2,3)\int \Phi_3 f(\mathcal{A}_2,\mathbf{B}_3)\,\mathrm{d}U_3. \tag{32}$$

To calculate the gauge transformation of the term $(3,1)(2,3)\int(\mathcal{A}_1,\mathbf{B}_2)\Phi_2^2\,\mathrm{d}U_2$, we use the equation $(2,3)\int(\mathcal{A}_1,\mathbf{B}_2)\Phi_2^2\,\mathrm{d}U_2 = (1,2)\int(\mathcal{A}_3,\mathbf{B}_2)\Phi_2^2\,\mathrm{d}U_2$, proved in Lemma 3.1. The gauge transformation follows:

$$(2,3)(3,1)\int \Phi_2^2(\mathcal{A}_1,\mathbf{B}_2)\,\mathrm{d}U_2 \rightsquigarrow (2,3)(3,1)\int \Phi_2^2(\mathcal{A}_1,\mathbf{B}_2)\,\mathrm{d}U_2 - 2(1,2)(1,3)(2,3)\int \Phi_2 f(\mathcal{A}_3,\mathbf{B}_2)\,\mathrm{d}U_2. \tag{33}$$

Note that the corresponding terms are cancelled. Namely, the additional term in (26) is transformed to $-2(2,3)^2\int(\operatorname{grad}\Phi_1, B_1)\Phi_1 f\,\mathrm{d}U_1$. This term cancels with the term in (28), because of the identity $\int \Phi_1^2(\operatorname{grad} f,\mathbf{B}_1)\,\mathrm{d}U_1 + 2\int(\operatorname{grad}\Phi_1,\mathbf{B}_1)\Phi_1 f\,\mathrm{d}U_1 = \int \operatorname{div}(\Phi_1^2 f\mathbf{B}_1)\,\mathrm{d}U_1 = 0$. The term in (25) cancels with the terms in (29) and (30), while the term in (27) cancels with the terms in (32) and (33).

Therefore the gauge transformation (24) leaves M unchanged. The invariance with respect to the gauge transformations

$$\mathcal{A}_2 \rightsquigarrow \mathcal{A}_2 + \operatorname{grad} f', \qquad \mathcal{A}_3 \rightsquigarrow \mathcal{A}_3 + \operatorname{grad} f''$$

are shown analogously.

Let us prove that for the gauge transformation (14), described in Lemma 2.5, preserve the invariant M. This transformation is presented as follows:

$$\varphi_{1,2} \rightsquigarrow \varphi_{1,2} + (3,1)C', \qquad \varphi_{2,1} \rightsquigarrow \varphi_{2,1} + (2,3)C',$$

if we put $C' = \mathbf{fl}_3 C$. The transformation of the functions Φ_1, Φ_2 is defined by the formula

$$\Phi_1 \rightsquigarrow \Phi_1 + (3,1)(2,3)C', \tag{34}$$

$$\Phi_2 \rightsquigarrow \Phi_2 - (2,3)(3,1)C'. \tag{35}$$

The transformation of the vector F, given by (20), follows:

$$F \rightsquigarrow F - (2,3)^2(3,1)C'\mathbf{B}_1 - (2,3)(3,1)^2 C'\mathbf{B}_2.$$

Recall that $(1,2) \neq 0$. For the transformation (14) we have

$$\varphi_{3,1} \rightsquigarrow \varphi_{3,1} - \frac{(2,3)(3,1)C'}{(1,2)}, \qquad \varphi_{3,2} \rightsquigarrow \varphi_{3,2} - \frac{(2,3)(3,1)C'}{(1,2)}.$$

In this case the functions Φ_1, Φ_2 are transformed with respect to Eqs. (34) and (35), and the function Φ_3 is unchanged. The vector F and the last term in (20) are transformed by the same formula and this proves the gauge invariance of M.

In the case $(1,2) = 0$, the proof of the gauge invariance follows from the results [16]. This case will be investigated in the next section because of the explicit example of the calculation. Herewith the gauge invariance of (20) is proved.

5. The invariant *M* is non-degenerated and cannot be expressed by the linking coefficients of the tubes

Let the field $\mathbf{B}$ be decomposed into three tubes with unit fluxes $\mathbf{fl}_1 = \mathbf{fl}_2 = \mathbf{fl}_3 = 1$. The tubes U_1 and U_2 are modelled on the Whitehead link, and the tube U_3 is arranged such that the pairs of tubes (U_1, U_3) and (U_2, U_3) present Hopf links with the linking coefficients +1 (see Fig. 2).

Because $(1,2) = 0$, $(2,3) = (3,1) = 1$, expressions (20) and (21) take the following form:

$$\mathbf{F} = \mathcal{A}_1 \times \mathcal{A}_2 - \psi_2 \mathbf{B}_1 + \psi_1 \mathbf{B}_2,$$

$$M(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) = \int [2\mathbf{G}\mathbf{F} - \psi_2^2(\mathcal{A}_1, \mathbf{B}_1) - \psi_1^2(\mathcal{A}_2, \mathbf{B}_2)]\, \mathrm{d}\mathbb{R}^3. \qquad (36)$$

This equation coincides with the integral formula for the Sato–Levine invariant presented in [15,16]. Because the Sato–Levine invariant for the Whitehead link is non-trivial (see [7]), we have $M \neq 0$. This proves that M is non-degenerated. If we change the pair of tubes U_1, U_2 to the trivial pair of tubes, keeping the pairs U_1, U_3 and U_2, U_3 in the isotopy class of the Hopf link, the value M becomes trivial. This proves that the invariant M cannot be expressed from the linking numbers of the components.

5.1. Problem

Is the invariant M a finite type invariant in the meaning of Vassiliev (see Remark 3.2)? Could we express this invariant from the Alexander polynomial?

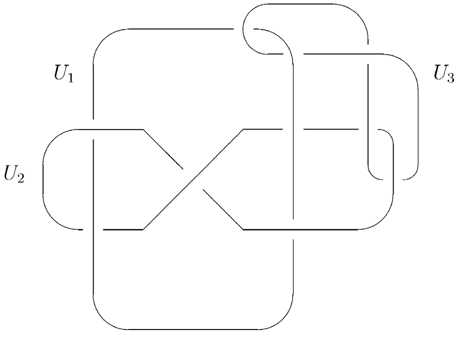

Fig. 2. A link with non-trivial *M*-invariant.

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